Logarithms

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Contents

1 History of Logarithms

The Scottish mathematician John Napier published his discovery of logarithms in his 1614 book titled *Mirifici Logarithmorum Canonis Descriptio.* His purpose was to assist in the multiplication of quantities that were then called sines. Since then, logarithms vastly reduced the time required for multiplying numbers with many digits. They were basic in numerical work for more than 300 years, until the perfection of mechanical calculating machines in the late 19th century and computers in the 20th century rendered them obsolete for large scale computations. The present-day notion of logarithms comes from Leonhard Euler, who connected them to the exponential function in the 18th century and also introduced the letter e as the base of natural logarithms. Logarithms are transcendental functions.

2 Introduction

A logarithm is the inverse of the exponential function. Specifically, a logarithm is the power to which a number(the base) must be raised to produce a given number.

> **Logarithm**
> Let a, b, c satisfy $a^c = b$, $a, b > 0$, and $a \neq 1$. Then, we define $\log_a b = c$. This can be read as "the logarithm of b with respect to the base a is equal to c".

There are three common particular bases in which logarithms are being represented which includes base 2 (binary logarithm),base 10(common logarithm) and base e(natural logarithms). Next, let's discuss the several identities of logs.

Theorem 2.1 (Logarithmic Identities)
The following are identities of logarithms that can be applied in solving problems;

1. $\log_a a = 1$
2. $\log_a 1 = 0$
3. $\log_a b + \log_a c = \log_a bc$
4. $\log_a b - \log_a c = \log_a \frac{b}{c}$
5. $\log_a b^c = c \log_a b$
6. $a^{\log_a b} = b$
7. $\log_a c \cdot \log_b d = \log_a d \cdot \log_b c$
8. $\log_{a^y} b^x = \frac{x}{y} \log_a b$
9. $\log_a b = \frac{\log_c b}{\log_c a}$ [This identity(Change of Base formula) has several consequences;
   - $\log_{a^b} c = \frac{1}{b} \log_a c$
   - $\log_a b = \frac{1}{\log_b a}$
   - $b^{\log_a d} = d^{\log_a b}$]
10. $\log_a(b + c) = \log_a b + \log_a(1 + \frac{c}{b})$,
11. $\log_a(b - c) = \log_a b + \log_a(1 - \frac{c}{b}), b > c$ (due to rounding errors,b and c may be switched if $c > b$)

Note $a, b, c,$ and d are positive real numbers while x and y are real numbers. Also,the above identities would not be proved here.

⚠ Be careful when working with logs, as the following equations are commonly mistaken and are NOT true in general.

$$\log_c (a + b) \neq \log_c a + \log_c b$$

$$\log_c(ab) \neq \log_c b \cdot \log_c a$$

$$(\log_c a)^n \neq \log_c a^n, n > 1$$

It's worth mentioning that in many scenarios, the base of a logarithm is not specified. In this case, it is assumed that the base is 10, a standard base that's chosen to align with our numeric system. Also, when you input log on a calculator, the function is the logarithm function with a base of 10. If you want to calculate a log with a different base on a calculator, then you can use the change of base formula to get everything in logs with base-10. For example, if you wanted to calculate $\log_3 5$, you would plug in $\frac{\log 5}{\log 3}$ on your calculator. Furthermore, the notation $ln(x)$ is shorthand notation for $\log_e x$, where $e \approx 2.71828$ is a mathematical constant. Why this constant is important is due to calculus: the exponential function e^x is equal to its own derivative. A final note is that a log that has no defined base, for example $\log(90)$, means that the log has a base of 10 in competition math. However, in more advanced math, a log without a base usually means a natural logarithm.

2.1 Sample Problems

Example 2.2
Compute the value of $\log_2 64$.

Solution. Let $\log_2 64 = x$. Then it suffices to find such x that satisfies $2^x = 64$. Short experimentation yields $x = 6$. Since, the logarithm is strictly increasing and injective, so there is only one value of x that works. □

Example 2.3 (2015 MAN Senior Mathematics Olympiad)
Find the value of x that satisfies

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40$$

Solution. We start by rewriting the equation above in exponential form

$$\log_{2^{\frac{1}{2}}} x^{\frac{1}{2}} + \log_2 x + \log_{2^2} x^2 + \log_{2^3} x^3 + \log_{2^4} x^4 = 40$$

Applying logarithmic identity(8) to each single term in the equation above yields :

$$\log_2 x + \log_2 x + \log_2 x + \log_2 x + \log_2 x = 40$$

$$= 5 \log_2 x = 40$$

$$= \log_2 x = 8$$

$$\Rightarrow x = 2^8 = \boxed{256}.$$

□

Example 2.4 (Brilliant.org)
What is(are) the solution(s) of the quadratic equation

$$\log_{10} 2x + \log_{10}(x-1) = \log_{10}(x^2+3)$$

Solution.

$$\log_{10} 2x + \log_{10}(x-1) = \log_{10}(x^2+3)$$

Applying the logarithmic identity(3) to the equation above yields:

$$\log_{10} 2x(x-1) = \log_{10}(x^2+3)$$

Since both of sides of the equation have the same logarithms, we equate both terms

$$2x(x-1) = (x^2+3)$$

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$x = -1, 3$$

Since the logarithm functions $\log(x-1)$ and $\log 2x$ are defined over positive integers, it must be true that $x - 1 > 0 \Rightarrow x > 1$ and $2x > 0 \Rightarrow x > 0$. Which implies $x \neq -1$ and $x = 3$. □

Example 2.5 (Math League 1988-1989)
What is the only real number $x > 1$ which satisfies the equation

$$\log_2 x \log_4 x \log_6 x = \log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x$$

Solution. We start by dividing both sides by $\log_2 x \log_4 x \log_6 x$ to get

$$\begin{aligned} 1 &= \frac{\log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x}{\log_2 x \log_4 x \log_6 x} \\ &= \frac{1}{\log_6 x} + \frac{1}{\log_4 x} + \frac{1}{\log_2 x} \\ &= \frac{1}{\log x / \log 6} + \frac{1}{\log x / \log 4} + \frac{1}{\log x / \log 2} \\ &= \frac{\log 6}{\log x} + \frac{\log 4}{\log x} + \frac{\log 2}{\log x} = \frac{\log 48}{\log x} \end{aligned}$$

Therefore $\log x = \log 48 \Rightarrow x = \boxed{48}$. □

Example 2.6 (AMC)
Suppose that $\log_{10} xy^3 = 1$ $\log_{10} x^2 y = 1$ What is $\log_{10} xy$?

Solution. Let's try to remove the logs from the equation using exponentiation, as logs are more complicated to deal with than exponents. This can be done by rewriting these equations into exponent form, giving us

$$\begin{aligned} 10 &= xy^3 \\ 10 &= x^2 y \end{aligned}$$

Now, how do we find xy from these equations? In order to find xy, we will need to multiply the equations in a way such that the exponent of x equals the exponent of y. This can be done by multiplying the square of the second equation with the first equation above to get $x^5y^5 = 10^3$. We can now convert this back to log form which results in $\log_{10} x^5y^5 = 3$. Furthermore, using logarithm identities, we have $\log_{10} x^5y^5 = \log_{10}(xy)^5 = 5\log_{10} xy$. Therefore, $\log_{10} xy = \frac{3}{5}$ □

Example 2.7 (1984 AIME problem 5)
Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

Solution. Adding the two equations and using the logarithmic identity (5) yields

$$\log_8 a + \log_8 b + \log_4 a^2 + log_4 b^2 = 12$$

$$\log_8(ab) + 2\log_4 ab = 12$$

Moreover, Since $\log_8 x = \frac{\log_2 x}{\log_2 8} = \frac{1}{3}\log_2 x$ and $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{1}{2}\log_2 x$, the above equation is equivalent to $\frac{4}{3}\log_2 ab = 12$. It follows that $\log_2 ab = 9$, and hence,

$$ab = 2^9 = \boxed{512}.$$

□

Example 2.8
Find $\log_{10} \frac{x}{y}$ if the following equations are true with $x > y$:

$$\log_{10} x + \log_{10} y = 10$$

$$\log_{10} x \cdot \log_{10} y = 16$$

Solution. Solving this problem seems messy at first, especially considering what the problem asks us to find! Let's try to simplify the problem by removing the logs. We can do this by substituting $a = \log_{10} x$ and $b = \log_{10} y$. We can then rewrite the two equations as $a + b = 10$ and $ab = 16$. Now, let's take a look at what the problem asks for. It asks for $\log_{10} \frac{x}{y}$, which we can rewritten as $\log_{10} x - \log_{10} y = a - b$ using the subtraction identity for logs. With this information, we now just need to find the value of $a - b$. We can square both sides of the equation $a + b = 10$ to get $a^2 + 2ab + b^2 = 100$. Since $ab = 16$, we can subtract $4ab = 64$ from both sides to get $a^2 - 2ab + b^2 = 36$. Now we can conveniently factor the left-hand side to get $(a - b)^2 = 36$. Since $x > y$, we have that $a > b$, so we take the positive solution of $a - b$. Hence, $a - b = \log_{10} \frac{x}{y} = 6$. □

3 Problems

Try to solve the following problems with the logarithmic identities you just learned.

Problem 1. Compute the value of $\log_3 81 - \log_3 27 + \log_3 243 + \log_3 6561$.

Problem 2. If $\log 3 = 0.4771, \log 5 = 0.6990$ and $\log 7 = 0.8451$. Evaluate $\log 105 + \log 15 + \log 35$.

Problem 3. Compute the value of $\log_4 8\sqrt{2} + \log_{25} 125\sqrt{5}$.

Problem 4. Given that $\log p = 2 \log x + 3 \log q$,express p in terms of x and q.

Problem 5. Find the positive value of x that satisfies $3 \log_8 x - \log_2 5 = \log_4(x + 6)$.

Problem 6 (AMC). The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

Problem 7. If $\log_7(\log_3(\log_2 x)) = 0$, find the value of $x^{-\frac{1}{2}}$.

Problem 8. Find n if $\log_{\log_2 n} n = 4$ and n is an integer.

Problem 9. Find $(\log_2 x)^2$if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

Problem 10 (2006 ARML Individual Problem 7). If $\log_8 a + \log_8 b = (\log_8 a) \cdot (\log_8 b)$ and $\log_a b = 3$, compute the value of a.

Problem 11. If $a^2 + b^2 = c^2$, then the value of $\frac{1}{\log_{c-a} b} + \frac{1}{\log_{c+a} b}$ is ?

Problem 12 (PuMaC 2019 Algebra B problem 2). If x is a real number so $3^x = 27x$, compute $\log_3(\frac{3^{3^x}}{x^{3^3}})$.

Problem 13 (2020 AIME II problem 3). The value of x that satisfies

$$\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$$

can be written as $\frac{m}{n}$ where m and n are relatively prime. Find $m + n$.

Problem 14 (1983 AIME problem 1). Let x, y and z all exceed 1 and let w be a positive number such that $\log_x w = 24, \log_y w = 40, \log_{xyz} w = 12$. Find $\log_z w$.

Problem 15. Find x if $log_{\frac{1}{\sqrt{2}}}(\frac{1}{\sqrt{8}}) = \log_2(4^x + 1) \cdot \log_2(4^{x+1} + 4)$.

Problem 16. If x, y, z and m, n, l are positive integers and the property

$$\frac{log_{10}x}{l + m - 2n} = \frac{log_{10}y}{m + n - 2l} = \frac{log_{10}z}{n + l - 2m}$$

holds. Then the product $x \cdot y \cdot z$ is ?

Problem 17 (HMMT). Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

Problem 18 (2013 AIME II problem 2). Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0$$

Find the sum of all possible values of $a + b$.

Problem 19. Simplify

$$\frac{1}{\log_2 A} + \frac{1}{\log_3 A} + \frac{1}{\log_4 A} + \ldots + \frac{1}{\log_{100} A}$$

where $A = (100!)^3$.

Problem 20. If

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \ldots + \frac{1}{\log_{2021} x} = 1$$

Find the largest prime factor of x.

Problem 21 (2005 ARML Team Question 3). Let A, R, M and L be positive real numbers such that $\log(A \cdot L) + \log(A \cdot M) = 2$, $\log(M \cdot L) + \log(M \cdot R) = 3$ and $\log(R \cdot A) + \log(R \cdot L) = 4$. Compute the value of the product $A \cdot R \cdot M \cdot L$.

Problem 22 (AIME II 2010/5). Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and

$$(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$$

Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

Problem 23 (AIME I 2020/2). There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 24 (AIME I 2012/9). Let x, y, and z be positive real numbers that satisfy

$$2 \log_x(2y) = 2 \log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 25. The value of x that satisfies

$$\log_{2021^{2x}} 2021^{2021} - \log_{2021} 2021^{2021} = 2021$$

can be written as $\frac{m}{n}$ where m and n are relatively prime. Find $m + n$.

Problem 26 (AIME). The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$

$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1y_1x_2y_2)$.