2021 Birthday Bash Mathematical Olympiad

Day 1, 19 July 2021

Problem 1. Find all triples x, y, z of non-negative integers satisfying $2^x = 3^y + 13^z$.

Problem 2. Let a, b and c be prime numbers such that $2abc + a + b + c = 2020$. Find the value of $ab + ac + bc$.

Problem 3. Triangle ABC has side lengths $AB = 13, AC = 14$ and $BC = 15$. A segment PQ is drawn parallel to BC and tangent to the incircle of triangle ABC with P on AB and Q on AC. If the distance from P to AC can be expressed as $\frac{a}{b}$ where $gcd(a, b) = 1$, find $a + b$.

Problem 4. Find in the closed form $1 \cdot 2\binom{n}{2} + 2 \cdot 3\binom{n}{3} + \cdots + (n-1) \cdot n\binom{n}{n}$.

Problem 5. Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfy

$$f(x^2 + f(x)f(y)) = xf(x+y)\forall x, y \in \mathbb{Z}.$$